Fermi National Accelerator Laboratory

Fermilab–Pub–89/192–T

Light-cone Recurrence Relations for QCD Amplitudes

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Abstract

I show how to calculate tree-level QCD multi-gluon amplitudes efficiently using a set of recurrence relations evaluated in the spinor-helicity basis of Xu, Zhang, and Chang.

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Operated by Universities Research Association Inc. under contract with the United States Department of Energy

1. Introduction

Perturbative calculations in quantum chromodynamics underlie most of our understanding of deep-inelastic lepton-hadron and hadron-hadron scattering. Parton-parton scattering amplitudes, combined with structure functions extracted from deep inelastic lepton-hadron scattering and with hadronization models (usually driven by Monte Carlo simulations) give predictions of multi-jet production at hadron-hadron colliders, which are important not only for testing QCD but also as calculations of backgrounds to new physics beyond the standard model.

One of the ingredients in this understanding is the perturbative calculation of parton-parton scattering amplitudes. Yet explicit calculations, even at tree-level, are technically complicated. Several groups have recently put forth approaches which serve to simplify the calculation of such amplitudes. Lee, Nair, and the author [1] have shown that one can calculate multi-gluon scattering amplitudes efficiently by considering them as the low-energy or infinite-tension limit of an open bosonic string theory. Mangano, Parke, and Xu [2] have shown that one can re-write tree-level Feynman diagrams for multi-gluon processes in a form which resembles that obtained in the string approach. Berends and Giele [3] have presented a recursive approach to this problem, in which the amplitude for an n-gluon process can be written in terms of known amplitudes for processes involving up to $n-1$ gluons. In all of these approaches, the full on-shell amplitude for an n-gluon process in an $SU(N)$ gauge theory can be written as a sum over non-cyclic permutations of the external legs,

$$\mathcal{A}_n(\{k_i,\varepsilon_i,a_i\}) = g^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}) A_n(k_{\sigma(1)},\varepsilon_{\sigma(1)};\ldots;k_{\sigma(n)},\varepsilon_{\sigma(n)}) \tag{1.1}$$

where k_i, ε_i, and a_i are respectively the momentum, polarization vector, and color index of the i-th external gluon. The T^a are the set of hermitian traceless $N\times N$ matrices (normalized so that $\mathrm{Tr}(T^aT^b)=\delta^{ab}$), and S_n/Z_n is the set of non-cyclic permutations of $\{1,\ldots,n\}$.

The ***partial amplitudes*** A_j possess a number of nice properties. Each is gauge invariant, that is invariant under the substitution $\varepsilon_i \to \varepsilon_i + \lambda k_i$ for each leg independently. It is also invariant under cyclic permutation of its arguments, and satisfies a reflection identity,

$$A(n,\ldots,1) = (-1)^n A_n(1,\ldots,n) \tag{1.2}$$

as well as a "twist" identity [1],

$$\sum_{\sigma\in Z_{n-1}} A_n(\sigma_1,\ldots,\sigma_{n-1},n) = 0. \tag{1.3}$$

(Mangano, Parke, and Xu [2] term the latter a dual Ward identity.) In addition, these amplitudes satisfy tree-level unitarity, which is to say they factorize on poles of a consecutive set of their arguments.

This representation has been extended to processes involving a single pair of massless quarks [4,3], and an arbitrary number of massless quarks [5, 6].

In the approach of refs. [1,6], QCD is embedded in an appropriate open string theory, and the field-theory amplitude emerges as the infinite-tension limit of the string amplitude. The properties of the field-theory partial amplitudes then emerge directly from known properties of the string-theory partial amplitudes [7].

Although an excursion through string theory makes clear the structure of the full amplitude as well as the symmetry properties of the partial amplitudes, it is not necessary for explicit calculations, as it is possible to formulate a diagrammatic expansion for the partial amplitudes that involves only field-theory propagators. I discuss this expansion, that of zero-mass mode or 'zero-mode' diagrams, and associated recurrence relations, in section 2. These recurrence relations are very similar in form to those of Berends and Giele [3]. The recurrence relations can be rephrased in the light-cone gauge, which, as discussed in section 3, is the natural gauge for using the spinor-helicity basis of of Xu, Zhang, and Chang [8], a convenient and efficient means of calculating explicit expressions for the partial amplitudes.

2. Zero-Mode Diagrams and Recurrence Relations

In an open string theory, the tree-level amplitudes can be written as a sum over non-cyclic permutations of partial amplitudes; this is of course the source of the representation (1.1) for gauge-theory amplitudes. The partial amplitudes can be written as an integral over Koba-Nielsen variables,

$$\begin{aligned} A_n(1,\cdots,n) = \mathcal{N}(\alpha')^{(n-4)/2} \int_0^1 dx_2\cdots dx_{n-2} \prod_{1\leq j<i\leq n} (x_i-x_j)^{\alpha' k_i\cdot k_j} \\ \times \exp\left(\sum_{j<i} \frac{\varepsilon_i\cdot\varepsilon_j}{(x_i-x_j)^2} - \sum_{i\neq j} \frac{\sqrt{\alpha'}k_i\cdot\varepsilon_j}{(x_i-x_j)}\right)\Bigg|_{\text{multilinear}} \end{aligned} \tag{2.1}$$

where $x_1 = 0$, $x_{n-1} = 1$, $x_n \to \infty$, and where the subscript 'multilinear' means that one should take only terms linear in all the polarization vectors when expanding the exponential. For $n > 4$, there are explicit powers of the inverse string tension α' in front of the integral, so any surviving contribution to the field-theory amplitudes comes from regions where the integral produces poles in α'. We may observe, however, that the α's inside the integral always appear in the combination

$\alpha' s_{ij}$, where $s_{ij} = 2k_i \cdot k_j$ is a momentum invariant, so poles in α' are equivalent to poles in the s_{ij}. Thus extracting the low-energy limit is equivalent to factorizing the amplitude on gluon poles. Unitarity (or explicit computation) tell us that we must sum over all possible factorizations on invariants built out of consecutive sets of external momenta. The four-point amplitude teaches us that in some terms, we should not factorize all the way: there are also contact terms coming from a four-gluon interaction. Thus we can build up the partial amplitudes out of vertices and propagators for the zero-mass modes of the string, since these are essentially what survive in the infinite-tension limit. This construction can be phrased in a diagrammatic manner, with rules analogous to those for conventional Feynman diagrams (the two kinds of diagrams are not interchangeable, however). The pseudo-Feynman rules for such gluonic zero-mode diagrams are as follows (the rules for including fermions have been presented elsewhere [6]; note that the normalizations are slightly different here). To compute a given partial amplitude, draw all planar diagrams using the vertices of fig. 1 and gluon propagators, with a specific cyclic ordering of the external legs. In the present case, where all the external legs are gluons, these partial amplitudes are the minimal gauge-invariant pieces to which one can attach a specified color factor.

All the external legs must satisfy the massless on-shell conditions $k^2 = 0$ and $k \cdot \varepsilon = 0$. In all the vertices given below, momentum conservation is enforced at every vertex, and all momenta are taken to be incoming. The appropriate internal vertices are given by removing the external wavefunction factors (the polarization vectors).

The three-gluon vertex (fig. 1a) represents a factor

$$\sqrt{2}i(k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 + k_3 \cdot \varepsilon_2 \, \varepsilon_1 \cdot \varepsilon_3 + k_1 \cdot \varepsilon_3 \, \varepsilon_1 \cdot \varepsilon_2) \tag{2.2}$$

where (for example) k_1 is the momentum flowing in through leg 1. This is in fact the same as the three-gluon vertex in the superstring theory, which is usually presented as an on-shell vertex; however, it is suitable for use as an off-shell vertex as well, since there are no associated Koba-Nielsen integrations. (The normalization of the vertex has been adjusted so that the with our normalization of the $SU(N)$ generators T^a, the coupling constant g in equation (1.1) is the same as the conventional coupling constant in QCD.)

The four-gluon vertex (fig. 1b) may be extracted from the infinite-tension limit of the four-gluon partial amplitude in the string theory; it is

$$i\varepsilon_1 \cdot \varepsilon_3 \, \varepsilon_2 \cdot \varepsilon_4 \tag{2.3}$$

Unlike the four-point Feynman vertex, this vertex is not gauge-invariant; the form given is the one in Feynman gauge.

Each gluon propagator connecting two vertices represents a conventional Feynman-gauge gluon propagator,

$$-ig_{\mu\nu}\frac{1}{k^2} \tag{2.4}$$

where k is the momentum flowing through the line.

The factorization naturally imposes a cyclic ordering on pieces of the amplitude; if, for example, we chop off a three-gluon vertex from the outside of a diagram, we will be left with a partial amplitude for a small number of legs, with one leg off-shell. We can use this view-point to build a recursive scheme for calculating the amplitudes, which is very similar to the recurrence scheme presented by Berends and Giele [3]; we must enumerate all possible factorizations involving three- and four-gluon vertices. Because of the presence of two types of vertices, the vertices should appear separately, in different terms; this in turn implies that the relations will involve factorization on two simultaneous poles to isolate the three-gluon vertices and factorization on three simultaneous poles to isolate four-gluon vertices.

Let us define partial amplitudes with polarization vector amputated,

$$A_{n,\mu}(1,\ldots,n) = \left[\frac{\partial}{\partial\varepsilon_n^\mu}A_n\right](1,\ldots,n) \tag{2.5}$$

sums of cyclicly consecutive momenta,

$$K_{i,j} = \begin{cases} \displaystyle\sum_{l=i}^{j} k_l & (i \le j) \\ \displaystyle-\sum_{l=j+1}^{i-1} k_l & (i > j) \end{cases} \tag{2.6}$$

and momentum invariants,

$$\begin{aligned} S_{i,i} &= 1 \\ S_{i,j} &= K_{i,j}^2 \qquad (i \neq j) \end{aligned} \tag{2.7}$$

The definition of $S_{i,i}$ allows us to treat the two-point amplitude on an equal footing with amplitudes for $n \geq 3$. For this purpose we also define

$$A_{2,\mu}(1,2) = i\varepsilon_{1\mu} \tag{2.8}$$

The recurrence relation for gluon amplitudes is then ($n \geq 3$)

$$
\begin{aligned}
A_{n+1}\ (k_1, \varepsilon_1; \ldots; k_n, \varepsilon_n; k_{n+1}, \varepsilon_{n+1}) = & \\
-\sum_{i=2}^{n} \frac{1}{S_{1,i-1}\, S_{i,n}} & A_{i,\mu}(k_1, \varepsilon_1; \ldots, k_{i-1}, \varepsilon_{i-1}; -K_{1,i-1}, 0) \\
& V_3^{\mu\nu\rho}(K_{1,i-1}; K_{i,n}; k_{n+1})\varepsilon_{n+1\,\rho} \\
& A_{n-i+2,\nu}(k_i, \varepsilon_i; \ldots; k_n, \varepsilon_n; -K_{i,n}, 0) \\
+ i \sum_{i=2}^{n-1} \sum_{j=2}^{n-i+1} & \frac{1}{S_{1,i-1}\, S_{i,i+j-2}\, S_{i+j-1,n}} V_4^{\mu\nu\rho\lambda} \\
& A_{i,\mu}(k_1, \varepsilon_1; \ldots; k_{i-1}, \varepsilon_{i-1}; -K_{1,i-1}, 0) \\
& A_{j,\nu}(k_i, \varepsilon_i; \ldots; k_{i+j-2}, \varepsilon_{i+j-2}; -K_{i,i+j-2}, 0) \\
& A_{n-i-j+3,\rho}(k_{i+j-1}, \varepsilon_{i+j-1}; \ldots; k_n, \varepsilon_n; -K_{i+j-1,n}, 0)\varepsilon_{n+1\,\lambda}
\end{aligned}
\tag{2.9}
$$

This relation correctly reproduces the four-gluon and five-gluon amplitudes analytically, and the correct six-gluon amplitudes [9, 2] numerically.

3. Light-Cone Gauge and the Spinor Helicity Basis

The recurrence relations of section 2 give a prescription for computing the amplitude as a formal polynomial in the polarization vectors and the momenta. But that is not really what we want. For explicit computations, we want the amplitude squared, generally summed over final-state colors and helicities, averaged over initial-state quantum numbers. The XZC spinor-helicity basis [8], along with the choices of reference momenta along the lines suggested by Mangano, Parke, and Xu [2] provide an efficient way of computing the various helicity amplitudes, and in turn, the amplitude squared, summed over helicities. Is there any way we can recast the recurrence relations to take advantage of the spinor-helicity formalism?

To investigate the possibility, let us consider structure of the gluon propagator. The recurrence relations of section 2 used differentiation with respect to the polarization vectors to obtain amputated amplitudes which were then tied together with propagators. Instead of amputating the amplitudes, however, we can also form the transverse projection tensor that is part of the vector propagator by summing over polarizations,

$$
\sum_i \varepsilon_\mu^{(i)}\, \varepsilon_\nu^{*(i)} = -g_{\mu\nu}
\tag{3.1}
$$

The gauge invariance of the partial amplitudes under gauge transformations, after suitable redefinitions of the intrinsic four-gluon coupling, allow us to choose a different gauge if we wish.

It turns out that for gluon amplitudes, light-cone gauge is the choice implemented by use of the spinor-helicity basis.

To see why this is so, let us compute

$$p_1 \cdot \sum_{\sigma=(+,-)} \varepsilon^{(\sigma)}(k;q)\varepsilon^{(-\sigma)}(-k;q) \cdot p_2 \tag{3.2}$$

Let us do this computation first in the case where all momenta are null vectors; then we will see how to fix this up in the general case. Using the spinor helicity basis definition of polarization vectors,

$$\begin{aligned}\varepsilon_\mu^{(+)}(k;q) &= \frac{\langle q_- | \gamma_\mu | k_- \rangle}{\sqrt{2}\,\langle q\, k\rangle} \\ \varepsilon_\mu^{(-)}(k;q) &= -\frac{\langle q_+ | \gamma_\mu | k_+ \rangle}{\sqrt{2}\,[q\, k]}\end{aligned} \tag{3.3}$$

and the explicit definition of the spinor product [8], we see that $\varepsilon(-k;q) = \varepsilon(k;q)$ (which is a reflection of C invariance), and our expression becomes

$$\frac{1}{2q \cdot k}\left(\langle q\,p_1\rangle\,[p_1\,k]\,\langle k\,p_2\rangle\,[p_2\,q] + \langle q\,p_2\rangle\,[p_2\,k]\,\langle k\,p_1\rangle\,[p_1\,q]\right) \tag{3.4}$$

where we use the notation

$$\begin{aligned}\langle i\,j\rangle &= \langle k_i\,k_j\rangle = \langle k_i^- \,|\, k_j^+\rangle \\ [i\,j] &= [k_i\,k_j] = \langle k_i^+ \,|\, k_j^-\rangle \\ (i\,j) &= 2\,k_i \cdot k_j = \langle i\,j\rangle\,[j\,i]\end{aligned} \tag{3.5}$$

Using Fierz-type identities [8] and antisymmetry of the spinor product, our expression becomes

$$\begin{aligned}&\frac{1}{2q \cdot k}\left(\langle q\,k\rangle\,\langle p_2\,p_1\rangle\,([p_1\,k]\,[q\,p_2] - [p_2\,k]\,[q\,p_1]) + \langle p_1\,k\rangle\,[p_1\,k]\,\langle q\,p_2\rangle\,[q\,p_2] + \langle p_2\,k\rangle\,[p_2\,k]\,\langle q\,p_1\rangle\,[q\,p_1]\right) \\ &= \frac{1}{2q \cdot k}\left(-\langle q\,k\rangle\,[q\,k]\,\langle p_2\,p_1\rangle\,[p_2\,p_1] + \langle p_1\,k\rangle\,[p_1\,k]\,\langle q\,p_2\rangle\,[q\,p_2] + \langle p_2\,k\rangle\,[p_2\,k]\,\langle q\,p_1\rangle\,[q\,p_1]\right)\end{aligned} \tag{3.6}$$

But $\langle p\,q\rangle\,[p\,q] = -2p \cdot q$, so this is simply

$$-p_1 \cdot p_2 + \frac{p_1 \cdot k\, q \cdot p_2 + p_1 \cdot q\, k \cdot p_2}{q \cdot k} = p_1 \cdot \Pi_{l.c.} \cdot p_2 \tag{3.7}$$

where the reference momentum q plays the role of the usual light-cone parameter n.

For our purposes, we must consider not only null vectors k and p_i, but also off-shell quantities. But every off-shell vector we encounter can be written as a sum of some of the external momenta, which are null vectors. Since any amplitude, as well as eqn. (3.2) is linear in the likes of p_1 and p_2, the extension to off-shell quantities is straightforward for them. Although the expression is also linear in k, each of the polarization vectors is not, so we must extend our definition of the spinor product to non-null vectors.

To do this, pick a standard basis for the momenta in the problem (e.g. eliminating the last momentum vector using momentum conservation), and express k as a sum of null vectors, $k = \sum_i q_i$. Each unit-helicity spinor of momentum k then carries an additional index i, that is, we turn it into a vector of unit-helicity spinors. By convention, bra vectors will carry an upper index, and ket vectors a lower index. When we multiply two polarization vectors (or for that matter any pair of spinor bra- and ket- vectors), we must also contract this additional index. The properties of spinor product generalize in a straightforward way; for example, it is still antisymmetric in the interchange of the two arguments, but because $\langle k\, k\rangle$ is a matrix rather than a scalar, antisymmetry does not force it to vanish. With these definitions, summing over the helicities $(\pm)$ produces the light-cone transverse projection operator for off-shell momenta as well.

In computing the partial amplitudes, we encounter this new vector index in both the numerator and denominator of expressions. In the numerator, we simply sum over it. In the denominator, we may observe that as we always multiply a positive-helicity polarization vector by a negative-helicity one carrying the same momentum, we will always end up with a conventional Lorentz inner product. So in the denominator of the polarization vectors, we may treat the spinor products for off-shell momenta simply as the square-root of the Lorentz product, ignoring the additional phase information present in the spinor products.

Thus for $q^2 = 0$ we define

$$
\begin{aligned}
\langle q\, k\rangle_* &= \begin{cases} \langle q\, k\rangle, & \text{if } k^2 = 0 \\ \sqrt{2q\cdot k}, & \text{if } k^2 \neq 0 \end{cases} \\
[q\, k]_* &= \begin{cases} [q\, k], & \text{if } k^2 = 0 \\ -\sqrt{2q\cdot k}, & \text{if } k^2 \neq 0 \end{cases}
\end{aligned}
\tag{3.8}
$$

which leads to the off-shell definitions for the polarization vectors

$$
\begin{aligned}
\varepsilon_\mu^{(+)}(\widehat{k};q) &= \frac{\langle q-|\,\gamma_\mu\,|k_-^i\rangle}{\sqrt{2}\langle q\, k\rangle_*} \\
\varepsilon_\mu^{(-)}(\widehat{k};q) &= -\frac{\langle q+|\,\gamma_\mu\,|k_+^i\rangle}{\sqrt{2}[q\, k]_*}
\end{aligned}
\tag{3.9}
$$

Several of the properties listed by Mangano et al. [2] for dot products of polarization vectors survive,

$$
\begin{aligned}
q\cdot\varepsilon^{(\pm)}(\widehat{k};q) &= 0 \\
\varepsilon^{(\pm)}(\widehat{k}_1;q)\cdot\varepsilon^{(\pm)}(\widehat{k}_2;q) &= 0 \\
\varepsilon^{(\pm)}(\widehat{k}_1;k_3)\cdot\varepsilon^{(\mp)}(k_3;q) &= 0
\end{aligned}
\tag{3.10}
$$

where the superscript $\widehat{\ }$ indicates an off-shell momentum.

In this framework, the recurrence relation eqn. (2.9) for gluon amplitudes becomes

$$
\begin{aligned}
A_{n+1}\,(k_1,\sigma_1;\ldots;k_n,\sigma_n;\widehat{k}_{n+1},\widehat{\sigma}_{n+1}) = \\
&-\sum_{i=2}^{n}\frac{1}{S_{1,i-1}\,S_{i,n}}\sum_{\widehat{\sigma}_x,\widehat{\sigma}_y=(+,-)} A_i(k_1,\sigma_1;\ldots,k_{i-1},\sigma_{i-1};-K_{1,i-1},\widehat{\sigma}_x) \\
&\qquad V_3(K_{1,i-1},\widehat{-\sigma_x};K_{i,n},\widehat{-\sigma_y};\widehat{k}_{n+1}\widehat{\sigma}_{n+1}) \\
&\qquad A_{n-i+2}(k_i,\sigma_i;\ldots;k_n,\sigma_n;-K_{i,n},\widehat{\sigma}_y) \\
&-i\sum_{i=2}^{n-1}\sum_{j=2}^{n-i+1}\frac{1}{S_{1,i-1}\,S_{i,i+j-2}\,S_{i+j-1,n}}\, V_4(\widehat{-\sigma_x},\widehat{-\sigma_y},\widehat{-\sigma_z},\widehat{\sigma}_{n+1}) \\
&\qquad \sum_{\widehat{\sigma}_x,\widehat{\sigma}_y,\widehat{\sigma}_z=(+,-)} A_i(k_1,\sigma_1;\ldots;k_{i-1},\sigma_{i-1};-K_{1,i-1},\widehat{\sigma}_x) \\
&\qquad A_j(k_i,\sigma_i;\ldots;k_{i+j-2},\sigma_{i+j-2};-K_{i,i+j-2},\widehat{\sigma}_y) \\
&\qquad A_{n-i-j+3}(k_{i+j-1},\sigma_{i+j-1};\ldots;k_n,\sigma_n;-K_{i+j-1,n},\widehat{\sigma}_z)
\end{aligned}
\tag{3.11}
$$

where the $\sigma_i = (+,-)$ are the helicities of the gluons, and where we have suppressed the additional index carried by the off-shell legs. (The minus sign in front of the four-gluon vertex terms is due to the minus sign in equation (3.1).)

The four-point coupling in this scheme is not gauge-independent; the correct form for light-cone gauge may be extracted from the four- and five-point functions,

$$
i\varepsilon_1\cdot\varepsilon_3\,\varepsilon_2\cdot\varepsilon_4 + i\varepsilon_1\cdot\varepsilon_2\,\varepsilon_3\cdot\varepsilon_4\,\frac{q\cdot(k_3-k_1)}{q\cdot(k_1+k_2)} + i\varepsilon_1\cdot\varepsilon_4\,\varepsilon_2\cdot\varepsilon_3\,\frac{q\cdot(k_4-k_2)}{q\cdot(k_2+k_3)} \tag{3.12}
$$

4. Applications

Using the the properties of products of polarization vectors we can use the light-cone recursion relations to compute various helicity amplitudes. We start by computing the three- and four-point vertices for various helicity configurations. Using eqn. (3.10b), we see immediately that

$$
\begin{aligned}
&V_3(\widehat{+}\widehat{+}\widehat{+}) = 0 \\
&V_4(\widehat{+}\widehat{+}\widehat{+}\widehat{+}) = 0 \\
&V_4(\widehat{+}\widehat{+}\widehat{+}\widehat{-}) = V_4(\widehat{-}\widehat{+}\widehat{+}\widehat{+}) = V_4(\widehat{+}\widehat{-}\widehat{+}\widehat{+}) = V_4(\widehat{+}\widehat{+}\widehat{-}\widehat{+}) = 0
\end{aligned}
\tag{4.1}
$$

We can also evaluate the other cases explicitly,

$$
\begin{aligned}
&V_3(\widehat{k}_1, \widehat{-}; \widehat{k}_2, \widehat{+}; \widehat{k}_3, \widehat{+}) = \\
&\qquad \frac{i\,\langle 1^i\, q\rangle}{[q\,1]_*\langle q\,2\rangle_*\langle q\,3\rangle_*}\left(\langle q^-|\,\not{k}_3\,|2_j{}^-\rangle\,[q\,3_k] + \langle q^-|\,\not{k}_1\,|3_k{}^-\rangle\,[q\,2_j]\right) \\
&V_4(\widehat{k}_1, \widehat{-}; \widehat{k}_2, \widehat{-}; \widehat{k}_3, \widehat{+}, \widehat{k}_4, \widehat{+}) = \\
&\qquad -\frac{i\,\langle 1^i\, q\rangle\,\langle 2^j\, q\rangle\,[q\,3_k]\,[q\,4_l]}{[q\,1]_*[q\,2]_*\langle q\,3\rangle_*\langle q\,4\rangle_*}\,\frac{q\cdot(k_1+k_2)}{q\cdot(k_2+k_3)} \\
&V_4(\widehat{k}_1, \widehat{-}; \widehat{k}_2, \widehat{+}; \widehat{k}_3, \widehat{-}, \widehat{k}_4, \widehat{+}) = \\
&\qquad \frac{i\,\langle 1^i\, q\rangle\,[q\,2_j]\,\langle 3^k\, q\rangle\,[q\,4_l]}{[q\,1]_*\langle q\,2\rangle_*[q\,3]_*\langle q\,4\rangle_*}\left(\frac{q\cdot(k_3-k_1)}{q\cdot(k_1+k_2)}+\frac{q\cdot(k_4-k_2)}{q\cdot(k_2+k_3)}\right)
\end{aligned}
\tag{4.2}
$$

where q is the light-cone parameter, and where we have written out explicitly the additional indices carried by off-shell spinors. Also, $A_2(\widehat{+}\widehat{+}) = 0$, while $A_2(\widehat{+}\widehat{-}) = -i$.

We can now derive some of the Parke-Taylor equations [10]. For example, leaving the momentum arguments implicit we have

$$
\begin{aligned}
&A_{n+1}(+\cdots+\widehat{+}) = \\
&\quad -\sum_{i=2}^{n}\frac{1}{S_{1,i-1}\,S_{i,n}}\sum_{\widehat{\sigma}_x,\widehat{\sigma}_y=(+,-)} A_i(+\cdots+\widehat{\sigma}_x)\,V_3(\widehat{-\sigma_x}\,\widehat{-\sigma_y}\,\widehat{+})\,A_{n-i+2}(+\cdots+\widehat{\sigma}_y) \\
&\quad -i\sum_{i=2}^{n-1}\sum_{j=2}^{n-i+1}\frac{1}{S_{1,i-1}\,S_{i,i+j-2}\,S_{i+j-1,n}} \\
&\qquad\qquad \sum_{\widehat{\sigma}_x,\widehat{\sigma}_y,\widehat{\sigma}_z=(+,-)} A_i(+\cdots+\widehat{\sigma}_x)A_j(+\cdots+\widehat{\sigma}_y)A_{n-i-j+3}(+\cdots+\widehat{\sigma}_z)V_4(\widehat{-\sigma_x}\,\widehat{-\sigma_y}\,\widehat{-\sigma_z}\,\widehat{+}) \\
&= -\sum_{i=2}^{n}\frac{1}{S_{1,i-1}\,S_{i,n}}\Big(A_i(+\cdots+\widehat{+})V_3(\widehat{-}\widehat{-}\widehat{+})A_{n-i+2}(+\cdots+\widehat{+}) \\
&\qquad\qquad + A_i(+\cdots+\widehat{+})V_3(\widehat{-}\widehat{+}\widehat{+})A_{n-i+2}(+\cdots+\widehat{-}) \\
&\qquad\qquad + A_i(+\cdots+\widehat{-})V_3(\widehat{+}\widehat{-}\widehat{+})A_{n-i+2}(+\cdots+\widehat{+})\Big) \\
&\quad -i\sum_{i=2}^{n-1}\sum_{j=2}^{n-i+1}\frac{1}{S_{1,i-1}\,S_{i,i+j-2}\,S_{i+j-1,n}} \\
&\qquad\qquad \Big(A_i(+\cdots+\widehat{+})A_j(+\cdots+\widehat{+})A_{n-i-j+3}(+\cdots+\widehat{-})V_4(\widehat{-}\widehat{-}\widehat{+}\widehat{+}) \\
&\qquad\qquad + A_i(+\cdots+\widehat{+})A_j(+\cdots+\widehat{-})A_{n-i-j+3}(+\cdots+\widehat{+})V_4(\widehat{-}\widehat{+}\widehat{-}\widehat{+}) \\
&\qquad\qquad + A_i(+\cdots+\widehat{-})A_j(+\cdots+\widehat{+})A_{n-i-j+3}(+\cdots+\widehat{+})V_4(\widehat{+}\widehat{-}\widehat{-}\widehat{+})\Big)
\end{aligned}
\tag{4.3}
$$

Every term on the right-hand side is proportional to $A_j(+\cdots+\hat{\mp})$ for some $j < n+1$, so we can show by induction that

$$A_n(+\cdots+\hat{\mp}) = 0 \tag{4.4}$$

by showing the equation holds true for one given value of n, say $n = 4$. But this case is simple, because $A_3(++\hat{\mp})$ and $A_2(\hat{\mp}\hat{\mp})$ both vanish trivially. This equation, evaluated on-shell for the last leg, gives us the first Parke-Taylor equation.

Using this equation, we also find

$$\begin{aligned} A_{n+1}\,(-+\cdots+\hat{\mp}) = \\ -\sum_{i=2}^{n}\frac{1}{S_{1,i-1}\,S_{i,n}}A_i(-+\cdots+\hat{\mp})V_3(\hat{\pm}\hat{\mp}\hat{\mp})A_{n-i+2}(+\cdots+\hat{-}) \end{aligned} \tag{4.5}$$

Once again, each term on the right-hand side is proportional to a similar amplitude for smaller n, and for $n = 4$,

$$A_4(-++\hat{\mp}) = \frac{i}{S_{2,3}}V_3(-\hat{\mp}\hat{\mp})A_3(++\hat{-}) + \frac{i}{S_{1,2}}A_3(-+\hat{\mp})V_3(\hat{\pm}+\hat{\mp}). \tag{4.6}$$

Now, we still have the freedom to choose the light-cone parameter q to be any null vector, in particular the momentum of the first gluon. In that case, we must choose a different reference momentum for the first leg, but that does not change the form of equations (4.5) and (4.6). With this choice, the right-hand side of equation (4.6) vanishes identically, and so again by induction,

$$A_n(-+\cdots+\hat{\mp})_{q=1} = 0 \tag{4.7}$$

which on-shell becomes the second Parke-Taylor equation.

The V_4 terms drop out of the recurrence relation for $A_n(++\cdots+\hat{-})$ as well, and we are left with

$$\begin{aligned} A_{n+1}(+\cdots+\hat{-}) &= \frac{i}{S_{2,n}}V_3(+\hat{\mp}_x\hat{-})A_n(+\cdots+\hat{-}_{(-x)}) \\ &\quad -\sum_{i=3}^{n}\frac{1}{S_{1,i-1}\,S_{i,n}}A_i(+\cdots+\hat{-}_{-y})V_3(\hat{\mp}_y\hat{\mp}_z\hat{-})A_{n-i+2}(+\cdots+\hat{-}_{(-z)}) \end{aligned} \tag{4.8}$$

where we have indicated the internal momenta by subscripts x, y, and z.

For A_3 and A_4, we find

$$A_3(+_1+_2\hat{-}_3) = -\frac{i\langle 3^l\, q\rangle \langle q\, 3\rangle_* k_3^2}{\langle 1\, 2\rangle \langle q\, 1\rangle \langle q\, 2\rangle}$$

$$\begin{aligned}
A_4(+_1+_2+_3\hat{-}_4) = &-\frac{i\langle 4^l\, q\rangle}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle q\, 1\rangle \langle q\, 3\rangle [q\, 4]_*} \\
&\left[\frac{\langle 1\, 2\rangle}{\langle q\, 2\rangle}\left(\langle q^-|\not{K}_{2,3}|1^-\rangle \langle q^-|\not{K}_{2,3}|q^-\rangle + \langle q^-|\not{k}_4\not{K}_{2,3}|q^-\rangle [q\, 1]\right)\right. \\
&\quad \left. + \frac{\langle 2\, 3\rangle}{\langle q\, 2\rangle}\left(\langle q\, 3\rangle \langle q^-|\not{K}_{1,2}|3^-\rangle [q\, 3] + \langle q^-|\not{k}_4|3^-\rangle \langle q^-|\not{K}_{1,2}|q^-\rangle\right)\right] \\
= &-\frac{i\langle 4^l\, q\rangle \langle q\, 4\rangle_* k_4^2}{\langle 1\, 2\rangle \langle 2\, 3\rangle \langle q\, 1\rangle \langle q\, 3\rangle}
\end{aligned} \tag{4.9}$$

where the off-shell index carried by the last leg is labelled by l. This leads to the ansatz

$$A_n(+\cdots+\hat{-}) = -\frac{i\langle n^l\, q\rangle \langle q\, n\rangle_* k_n^2}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle (n-2)\, (n-1)\rangle \langle q\, 1\rangle \langle q\, (n-1)\rangle} \tag{4.10}$$

which we prove by induction:

$$\begin{aligned}
&A_{n+1}(+\cdots+\hat{-}) = \\
&\quad \frac{i}{S_{2,n}} V_3(+\hat{\mp}_x\hat{-}_{n+1}) A_n(+\cdots+\hat{-}_{(-x)}) + \frac{i}{S_{1,n-1}} A_n(+\cdots+\hat{-}_{(-y)}) V_3(\hat{\mp}_y+\hat{-}_{n+1}) \\
&\quad - \sum_{j=3}^{n-1} \frac{1}{S_{1,j-1} S_{j,n}} A_j(+\cdots+\hat{-}_{(-z(j))}) V_3(\hat{\mp}_{z(j)}\hat{\mp}_{t(j)}\hat{-}) A_{n-j+2}(+\cdots+\hat{-}_{(-t(j))}) \\
&= -\frac{i\langle (n+1)^l\, q\rangle}{[q\, (n+1)]_* \langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle (n-1)\, n\rangle \langle q\, 1\rangle \langle q\, n\rangle} \\
&\quad \left[\frac{\langle 1\, 2\rangle}{\langle q\, 2\rangle} \langle q^-|\not{k}_{n+1}|1^-\rangle\, q\cdot k_{n+1} + \frac{\langle (n-1)\, n\rangle}{\langle q\, (n-1)\rangle} \langle q^-|\not{k}_{n+1}|n^-\rangle\, (-1) q\cdot k_{n+1}\right. \\
&\quad \left. - \sum_{j=3}^{n-1} \frac{\langle (j-1)\, j\rangle}{\langle q\, (j-1)\rangle \langle q\, j\rangle} \langle q^-|\not{k}_{n+1}\not{K}_{j,n}|q^+\rangle\, (-1) q\cdot k_{n+1}\right] \\
&= \frac{i\langle (n+1)^l\, q\rangle \langle q\, (n+1)\rangle_*}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle (n-1)\, n\rangle \langle q\, 1\rangle \langle q\, n\rangle} \sum_{j=2}^{n} \frac{\langle (j-1)\, j\rangle}{\langle q\, (j-1)\rangle \langle q\, j\rangle} \langle q^-|\not{k}_{n+1}\not{K}_{j,n}|q^+\rangle \\
&= \frac{i\langle (n+1)^l\, q\rangle \langle q\, (n+1)\rangle_*}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle (n-1)\, n\rangle \langle q\, 1\rangle \langle q\, n\rangle} \langle q\, (n+1)_m\rangle \sum_{j=2}^{n} \frac{\langle (j-1)\, j\rangle}{\langle q\, (j-1)\rangle \langle q\, j\rangle} \langle q^-|\not{K}_{j,n}|(n+1)_m{}^-\rangle
\end{aligned} \tag{4.11}$$

Using the identity

$$\frac{\langle (j-1)\, j\rangle}{\langle q\, (j-1)\rangle \langle q\, j\rangle} + \frac{\langle (j-2)\, (j-1)\rangle}{\langle q\, (j-2)\rangle \langle q\, (j-1)\rangle} = \frac{\langle (j-2)\, j\rangle}{\langle q\, (j-2)\rangle \langle q\, j\rangle} \tag{4.12}$$

which follows from a Fierz identity, we have

$$\sum_{j=m}^{n} \frac{\langle (j-1)\, j\rangle}{\langle q\, (j-1)\rangle \langle q\, j\rangle} = \frac{\langle (m-1)\, n\rangle}{\langle q\, (m-1)\rangle \langle q\, n\rangle} \tag{4.13}$$

and

$$\begin{aligned}
&\sum_{j=m}^{n} \frac{\langle (j-1)\, j\rangle}{\langle q\,(j-1)\rangle\,\langle q\, j\rangle}\,\langle q^-|\,\not{K}_{j,n}\,|p^-\rangle \\
&\quad = \sum_{j=m}^{n}\sum_{r=j}^{n} \frac{\langle (j-1)\, j\rangle}{\langle q\,(j-1)\rangle\,\langle q\, j\rangle}\,\langle q\, r\rangle\,[r\, p] \\
&\quad = \sum_{r=m}^{n}\sum_{j=m}^{r} \frac{\langle (j-1)\, j\rangle}{\langle q\,(j-1)\rangle\,\langle q\, j\rangle}\,\langle q\, r\rangle\,[r\, p] \\
&\quad = \sum_{r=m}^{n} \frac{\langle (m-1)\, r\rangle}{\langle q\,(m-1)\rangle\,\langle q\, r\rangle}\,\langle q\, r\rangle\,[r\, p] \\
&\quad = \frac{\langle (m-1)^-|\,\not{K}_{m,n}\,|p^-\rangle}{\langle q\,(m-1)\rangle}
\end{aligned} \tag{4.14}$$

The expression for A_{n+1} then becomes

$$\begin{aligned}
&\frac{i\,\langle (n+1)^l\, q\rangle\,\langle q\,(n+1)\rangle_*}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\, n\rangle\,\langle q\,1\rangle\,\langle q\, n\rangle}\,\langle q\,(n+1)_m\rangle\,\frac{\langle 1^-|\,\not{K}_{2,n}\,|(n+1)_m{}^-\rangle}{\langle q\,1\rangle} \\
&= -\frac{i\,\langle (n+1)^l\, q\rangle\,\langle q\,(n+1)\rangle_*}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-1)\, n\rangle\,\langle q\,1\rangle\,\langle q\, n\rangle}\,\frac{1}{\langle q\,1\rangle}\,\langle q^-|\,\not{k}_{n+1}\not{k}_{n+1}\,|1^+\rangle \\
&= -\frac{i\,\langle (n+1)^l\, q\rangle\,\langle q\,(n+1)\rangle_* k_{n+1}^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle q\,1\rangle\,\langle q\, n\rangle}
\end{aligned} \tag{4.15}$$

Similarly, for $A_n(- + \cdots + \hat{\mp})$, where the light-cone parameter is not the momentum of the first gluon, we find

$$\frac{i\,\langle 1^-|\,\not{k}_n\,|n_l{}^-\rangle\,\langle q\,1\rangle^2}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle (n-2)\,(n-1)\rangle\,\langle q\,(n-1)\rangle\,\langle q\, n\rangle_*} \tag{4.16}$$

To derive the final Parke-Taylor equation, we will also need

$$V_3(-_1\hat{-}_2\hat{\mp}_3)_{q=1} = \frac{i\,\langle 2^j\,1\rangle\,\langle 1^-|\,\not{k}_2\,|r^-\rangle\,[1\,3_k]}{[1\, r]\,[1\,2]_*\langle 1\,3\rangle_*} \tag{4.17}$$

where the subscript $q = 1$ indicates that the light-cone parameter is the momentum of the first external gluon, and where r is the reference momentum for that gluon.

Choosing the second external gluon to be the reference momentum for the first external gluon gives

$$A_3(--\hat{\mp})_{q=1,r=2}=0$$

$$A_4(--+\hat{\mp})_{q=1,r=2}=\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 1\,3\rangle\,\langle 1\,4\rangle_*}\left(\frac{\langle 1\,3\rangle\,[3\,4_k]}{S_{3,4}}+\frac{\langle 1^-|\,\not{k}_4\,|4_k{}^-\rangle}{(1\,4)}+\frac{\langle 1^-|\,\not{k}_4\,|4_k{}^-\rangle}{S_{3,4}}\right)$$

$$A_5(--++\hat{\mp})_{q=1,r=2}=\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 1\,4\rangle\,\langle 1\,5\rangle_*}$$
$$\times\left(\frac{\langle 1\,4\rangle\,[4\,5_k]}{S_{4,5}}+\frac{\langle 1^-|\,\not{k}_5\,|5_k{}^-\rangle}{(1\,5)}+\frac{\langle 1^-|\,\not{k}_5\,|5_k{}^-\rangle}{S_{4,5}}+\frac{\langle 1^-|\,\not{k}_2\not{k}_3\not{k}_5\,|5_k{}^-\rangle}{S_{1,2}S_{1,3}}\right)$$

$$A_6(--+++\hat{\mp})_{q=1,r=2}=\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 1\,5\rangle\,\langle 1\,6\rangle_*}$$
$$\times\left(\frac{\langle 1\,5\rangle\,[5\,6_k]}{S_{5,6}}+\frac{\langle 1^-|\,\not{k}_6\,|6_k{}^-\rangle}{(1\,6)}+\frac{\langle 1^-|\,\not{k}_6\,|6_k{}^-\rangle}{S_{5,6}}\right.$$
$$\left.+\frac{\langle 1^-|\,\not{k}_2\not{k}_3\not{k}_6\,|6_k{}^-\rangle}{S_{1,2}S_{1,3}}+\frac{\langle 1^-|\,\not{K}_{2,3}\not{k}_4\not{k}_6\,|6_k{}^-\rangle}{S_{1,3}S_{1,4}}\right) \tag{4.18}$$

This leads to the ansatz

$$A_n(--+\cdots+\hat{\mp})_{q=1,r=2}=$$
$$\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle (n-2)\,(n-1)\rangle\,\langle 1\,(n-1)\rangle\,\langle 1\,n\rangle_*}$$
$$\times\left(\frac{\langle 1\,(n-1)\rangle\,[(n-1)\,n_k]}{S_{n-1,n}}+\frac{\langle 1^-|\,\not{k}_n\,|n_k{}^-\rangle}{(1\,n)}+\frac{\langle 1^-|\,\not{k}_n\,|n_k{}^-\rangle}{S_{n-1,n}}+\sum_{l=3}^{n-2}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{k}_n\,|n_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right) \tag{4.19}$$

which we prove by calculating A_{n+1}:

$$A_{n+1}(--+\cdots+\hat{\mp})_{q=1,r=2}=$$
$$\frac{i}{S_{1,n}}V_3(-\hat{-}_x\hat{\mp}_{n+1})A_n(-+\cdots+\hat{\mp}_{-x})$$
$$-\sum_{j=4}^{n-1}\frac{1}{S_{1,j-1}S_{j,n}}A_j(--+\cdots+\hat{\mp}_{-y(j)})V_3(\hat{-}_{y(j)}\hat{\mp}_{z(j)}\hat{\mp}_{n+1})A_{n-j+2}(+\cdots+\hat{-}_{(-z(j))})$$
$$+\frac{i}{S_{1,n-1}}A_n(--+\cdots+\hat{\mp}_{-y(n)})V_3(\hat{-}_{y(n)}+_n\hat{\mp}_{n+1})$$
$$=\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)}-\frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.$$
$$-\sum_{j=4}^{n}\frac{\langle (j-1)\,j\rangle\,\langle 1^-|\,\not{K}_{2,j-2}\,|(j-1)^-\rangle\,\langle 1^-|\,\not{K}_{j,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,j\rangle\;S_{1,j-2}S_{1,j-1}}$$
$$\left.+\sum_{j=4}^{n}\frac{\langle (j-1)\,j\rangle\,\langle 1^-|\,\not{K}_{j,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(j-1)\rangle\,\langle 1\,j\rangle}\sum_{l=3}^{j-2}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|1^+\rangle}{S_{1,l-1}S_{1,l}}\right)$$

$$
\begin{aligned}
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.\\
&\quad - \sum_{l=3}^{n-1}\frac{\langle l\,(l+1)\rangle\,\langle 1^-|\,\not{K}_{2,l-1}\,|l^-\rangle\,\langle 1^-|\,\not{K}_{l+1,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(l+1)\rangle\,S_{1,l-1}S_{1,l}}\\
&\quad \left. + \sum_{l=3}^{n-2}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|1^+\rangle}{S_{1,l-1}S_{1,l}}\sum_{j=l+2}^{n}\frac{\langle (j-1)\,j\rangle\,\langle 1^-|\,\not{K}_{j,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(j-1)\rangle\,\langle 1\,j\rangle}\right)
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.\\
&\quad - \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|(l+1)^+\rangle\,\langle 1^-|\,\not{K}_{l+1,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(l+1)\rangle\,S_{1,l-1}S_{1,l}}\\
&\quad \left. + \sum_{l=3}^{n-2}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|1^+\rangle\,\langle (l+1)^-|\,\not{K}_{l+2,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(l+1)\rangle\,S_{1,l-1}S_{1,l}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.\\
&\quad - \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|(l+1)^+\rangle\,\langle 1^-|\,\not{K}_{l+1,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(l+1)\rangle\,S_{1,l-1}S_{1,l}}\\
&\quad \left. + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\,|1^+\rangle\,\langle (l+1)^-|\,\not{K}_{l+1,n}\,|(n+1)_k{}^-\rangle}{\langle 1\,(l+1)\rangle\,S_{1,l-1}S_{1,l}}\right)
\end{aligned}
$$

We can combine corresponding terms in the two sums using a Fierz identity to obtain

$$
\begin{aligned}
&\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.\\
&\quad \left. - \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{K}_{l+1,n}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right)\\
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*}\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}}\right.\\
&\quad - \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{K}_{1,l}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\\
&\quad \left. + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right)
\end{aligned}
\tag{4.21}
$$

Rearranging $\not{k}$s gives

$$
\begin{aligned}
&\frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*} \\
&\quad\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1\,2\rangle\,[2\,(n+1)_k]}{S_{1,2}} + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{k}_l\,|(n+1)_k{}^-\rangle}{S_{1,l}}\right. \\
&\quad\left. -\sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\,|(n+1)_k{}^-\rangle\,(S_{1,l}-S_{1,l-1})}{S_{1,l-1}S_{1,l}} + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right) \\
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*} \\
&\quad\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} - \frac{\langle 1^-|\,\not{k}_{2,n-1}\,|(n+1)_k{}^-\rangle}{S_{1,n-1}} + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right) \\
&= \frac{i\langle 1\,2\rangle^3}{\langle 2\,3\rangle\cdots\langle (n-1)\,n\rangle\,\langle 1\,n\rangle\,\langle 1\,(n+1)\rangle_*} \\
&\quad\left(\frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{(1\,n+1)} + \frac{\langle 1\,n\rangle\,[n\,(n+1)_k]}{S_{n,n+1}} + \frac{\langle 1^-|\,\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{S_{n,n+1}}\right. \\
&\qquad\left. + \sum_{l=3}^{n-1}\frac{\langle 1^-|\,\not{K}_{2,l-1}\not{k}_l\not{k}_{n+1}\,|(n+1)_k{}^-\rangle}{S_{1,l-1}S_{1,l}}\right)
\end{aligned}
\tag{4.22}
$$

which is the desired result.

If we take the last leg on-shell, we obtain the third Parke-Taylor equation,

$$
A_n(--+\cdots+) = \frac{i\,\langle 1\,2\rangle^3}{\langle 2\,3\rangle\,\langle 3\,4\rangle\cdots\langle (n-1)\,n\rangle\,\langle n\,1\rangle}. \tag{4.23}
$$

5. New Results

The recursion relations can be used to derive compact expressions for other helicity amplitudes as well; off-shell expressions, such as those presented in the previous section, and those we shall present below, can be used to speed up the numerical evaluation of helicity amplitudes for which compact analytic expressions are not available, while on-shell expressions can be used directly.

After an appropriate amount of algebra, one finds and proves the correctness of ansätze for

$A_n(+\cdots+-_m+\cdots+\widehat{\mp})$, $A_n(-+\cdots+\widehat{-})$, and $A_n(+\cdots+-\widehat{-})$ $(n\geq 4)$:

$$
\begin{aligned}
A_n(+\cdots+-_m+\cdots+\widehat{\mp}) &= \frac{i\langle q\,m\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-2)\,(n-1)\rangle\langle q\,1\rangle\langle q\,(n-1)\rangle\langle q\,n\rangle_*} \\
&\times\Bigg(\langle m^-|\not{k}_n|n_j{}^-\rangle - \frac{\langle q^-|\not{k}_n|n_j{}^-\rangle\langle m^-|\not{K}_{1,m}|q^-\rangle}{(q\,K_{m,n-1})} - \langle q^-|\not{k}_n|n_j{}^-\rangle[q\,m]\sum_{l=1}^{m-1}\frac{[q\,l]\langle l^-|\not{K}_{1,l}|q^-\rangle}{(q\,K_{l,n-1})(q\,K_{l+1,n-1})}\Bigg) \\
A_n(-+\cdots+\widehat{-}) &= \frac{i}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-2)\,(n-1)\rangle\langle q\,(n-1)\rangle[q\,n]_*} \\
&\times\Bigg(-\frac{\langle n^j\,1\rangle\langle 1^-|\not{k}_n|q^-\rangle\langle q\,n-1\rangle\langle 1^-|\not{k}_n|n-1^-\rangle}{S_{n-1,n}} - \frac{\langle n^j\,1\rangle\,k_n^2\langle 1^-|\not{k}_n|q^-\rangle\langle q\,1\rangle}{S_{n-1,n}} \\
&\quad -\frac{\langle n^{j-}|\not{k}_n\not{k}_{n-1}|1^+\rangle\langle 1^-|\not{k}_n|q^-\rangle\langle q\,1\rangle}{S_{n-1,n}} + \langle n^j\,q\rangle(q\,n)\,k_n^2\langle q\,1\rangle\frac{\langle 1^-|\not{K}_{1,2}|q^-\rangle\langle 1^-|\not{K}_{1,2}|q^-\rangle}{(q\,1)(q\,K_{1,2})\,S_{1,2}} \\
&\quad -\langle n^j\,q\rangle(q\,n)\,k_n^2\sum_{l=3}^{n-2}\frac{\langle 1^-|\not{K}_{1,l}|q^-\rangle\langle 1^-|\not{K}_{1,l-1}\not{k}_l|1^+\rangle}{(q\,K_{1,l})\,S_{1,l-1}S_{1,l}} \\
&\quad +\langle n^j\,q\rangle(q\,n)\,k_n^2\langle q\,1\rangle\sum_{l=3}^{n-1}\frac{\langle 1^-|\not{K}_{1,l}|q^-\rangle\langle 1^-|\not{K}_{1,l-1}|q^-\rangle}{(q\,K_{1,l-1})(q\,K_{1,l})\,S_{1,l-1}}\Bigg) \\
A_n(+\cdots+-\widehat{-}) &= \frac{i}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-2)\,(n-1)\rangle\langle q\,1\rangle[q\,n]_*} \\
&\times\Bigg(-\frac{\langle n^j\,(n-1)\rangle\langle (n-1)^-|\not{k}_n|q^-\rangle\langle q\,1\rangle\langle (n-1)^-|\not{k}_n|1^-\rangle}{S_{2,n-1}} \\
&\quad +\frac{\langle n^j\,(n-1)\rangle\,k_n^2\langle (n-1)^-|\not{K}_{2,n-1}|q^-\rangle\langle q\,(n-1)\rangle}{S_{2,n-1}} \\
&\quad -\frac{\langle n^{j-}|\not{k}_n\not{k}_1|(n-1)^+\rangle(q\,n)\langle (n-1)^-|\not{K}_{2,n-1}|q^-\rangle\langle q\,(n-1)\rangle}{S_{2,n-1}\,(q\,K_{2,n-1})} \\
&\quad +\frac{\langle n^j\,q\rangle\;k_n^2\;(q\,n)\langle q\,(n-1)\rangle\langle (n-1)^-|\not{k}_{n-2}|q^-\rangle^2}{S_{n-2,n-1}\,(q\,K_{n-2,n-1})(q\,k_{n-1})} \\
&\quad -\langle n^j\,q\rangle\,k_n^2\,(q\,n)\sum_{l=3}^{n-2}\frac{\langle (n-1)^-|\not{K}_{l-1,n-1}|q^-\rangle\langle (n-1)^-|\not{K}_{l,n-1}\not{k}_{l-1}|(n-1)^+\rangle}{S_{l,n-1}S_{l-1,n-1}\,(q\,K_{l-1,n-1})} \\
&\quad +\langle n^j\,q\rangle\,k_n^2\,(q\,n)\langle q\,(n-1)\rangle\sum_{l=3}^{n-2}\frac{\langle (n-1)^-|\not{k}_{l-1}|q^-\rangle\langle (n-1)^-|\not{K}_{l,n-1}|q^-\rangle}{S_{l,n-1}\,(q\,K_{l-1,n-1})(q\,K_{l,n-1})} \\
&\quad -\langle n^{j-}|\not{k}_n|q^-\rangle(q\,n)\langle q\,(n-1)\rangle^2\sum_{l=3}^{n-1}\frac{\langle (n-1)^-|\not{k}_{l-1}|q^-\rangle}{(q\,K_{l-1,n-1})(q\,K_{l,n-1})}\Bigg)
\end{aligned}
\tag{5.1}
$$

With these off-shell amplitudes, we can solve for one partial amplitude with three opposite-helicity gluons, $A_n(-+\cdots+--)$, without having to guess an ansatz. It is convenient to choose the light-cone parameter to be the momentum of the last gluon, and the latter's reference momentum to be the momentum of the first gluon; then, $V(\widehat{\mp}\widehat{\mp}\widehat{-}) = 0$, and the recursion relation (3.11) becomes

$$
\begin{aligned}
&A_n(-+\cdots+--) = \\
&\quad -\sum_{l=2}^{n-1} \frac{1}{S_{1,l-1}S_{l,n-1}} A_l(-+\cdots+\widehat{\mp})_{q=n} V(\widehat{-}\widehat{-}\widehat{-})_{q=n,r_n=1} A_{n-l+1}(+\cdots+-\widehat{\mp})_{q=n} \\
&\quad -\sum_{l=2}^{n-1} \frac{1}{S_{1,l-1}S_{l,n-1}} A_l(-+\cdots+\widehat{-})_{q=n} V(\widehat{\mp}\widehat{-}\widehat{-})_{q=n,r_n=1} A_{n-l+1}(+\cdots+-\widehat{\mp})_{q=n} \\
&\quad -\sum_{l=2}^{n-1} \frac{1}{S_{1,l-1}S_{l,n-1}} A_l(-+\cdots+\widehat{\mp})_{q=n} V(\widehat{-}\widehat{\mp}\widehat{-})_{q=n,r_n=1} A_{n-l+1}(+\cdots+-\widehat{-})_{q=n} \\
&\quad -\sum_{l=2}^{n-2}\sum_{m=2}^{n-l} A_l(-+\cdots+\widehat{\mp})_{q=n} A_m(+\cdots+\widehat{-})_{q=n} V(\widehat{-}\widehat{\mp}\widehat{-}\widehat{-})_{q=n,r_n=1} A_{n-l-m+2}(+\cdots+-\widehat{\mp})_{q=n} \\
&= \frac{i}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle (n-2)\,(n-1)\rangle\,[(n-2)\,(n-1)]\,[(n-1)\,n]\,[n\,1]\,[1\,2]} \\
&\quad\times\Bigg(\frac{(n-1\,n)\langle 1\,2\rangle\,[(n-1)\,(n-2)]\,\langle (n-1)^-|\,K_-\,|2^-\rangle^2}{S_{3,n-1}} + \frac{(1\,n)\,\langle (n-1)\,(n-2)\rangle\,[1\,2]\,\langle 1^-|\,K_-\,|(n-2)^-\rangle^2}{S_{1,n-3}} \\
&\quad + \frac{\langle n\,(n-1)\rangle\langle 1\,(n-1)\rangle\langle 1\,(n-2)\rangle\,[1\,n]\,[1\,2]\,[(n-1)\,(n-2)]\,\langle 1^-|\,K_-\,|(n-2)^-\rangle}{S_{1,n-3}} \\
&\quad + \frac{\langle n\,1\rangle\langle (n-1)\,1\rangle\langle (n-1)\,2\rangle\,[(n-1)\,n]\,[1\,2]\,[(n-1)\,(n-2)]\,\langle (n-1)^-|\,K_-\,|2^-\rangle}{S_{3,n-1}} \\
&\quad - \langle 1\,(n-1)\rangle^2 S_{2,n-2}\,[1\,2]\,[(n-1)\,(n-2)] - \frac{(n-1\,n)\,(1\,n)\,\langle 1\,(n-1)\rangle\,\langle 1^-|\,K_-\,|(n-2)^-\rangle\,[1\,2]}{S_{1,n-3}} \\
&\quad - [n\,1]\,[n\,(n-1)]\,[1\,2]\,[(n-1)\,(n-2)] \\
&\qquad \times\sum_{l=3}^{n-3}\Bigg[\frac{\langle n\,(n-1)\rangle^2\langle (n-1)\,1\rangle\,\langle 1^-|\,\not{K}_{1,l-1}\not{k}_l\,|1^+\rangle}{S_{1,l-1}S_{1,l}} \\
&\qquad + \frac{\langle n\,1\rangle^2\langle 1\,(n-1)\rangle\,\langle (n-1)^-|\,\not{K}_{l+1,n-1}\not{k}_l\,|(n-1)^+\rangle}{S_{l+1,n-1}S_{l,n-1}} \\
&\qquad - \frac{\langle n\,1\rangle\langle n\,(n-1)\rangle\langle (n-1)\,1\rangle\,\langle (n-1)^-|\,\not{K}_{l+1,n}\not{k}_l\,|1^+\rangle}{S_{1,l}S_{l,n-1}} \\
&\qquad - \frac{\langle n\,1\rangle\langle n\,(n-1)\rangle^2\,\langle (n-1)^-|\,\not{K}_{l+1,n}\not{k}_l\,|1^+\rangle\,\langle 1^-|\,\not{K}_{l,n}\,|n^-\rangle}{S_{1,l-1}S_{1,l}S_{l,n-1}} \\
&\qquad - \frac{\langle n\,1\rangle^2\langle n\,(n-1)\rangle\,\langle (n-1)^-|\,\not{K}_{l+1,n-1}\not{k}_l\,|1^+\rangle\,\langle (n-1)^-|\,\not{K}_{l+1,n}\,|n^-\rangle}{S_{1,l}S_{l+1,n-1}S_{l,n-1}} \Bigg]\Bigg)
\end{aligned}
\tag{5.2}
$$

where $K_- = k_1 + k_{n-1} + k_n$. While this form is not quite manifestly symmetric (for even n) or antisymmetric (for odd n), as required by equation (1.2), it in fact does possess the required symmetry. This expression is more complicated than the Parke-Taylor equation for amplitudes with two opposite helicities, but one may observe two features: (a) the number of terms increases only linearly with n, rather than factorially as might be feared from the Feynman diagram expansion, and (b) most of the two-particle singularities are concentrated in the common denominator outside the parentheses.

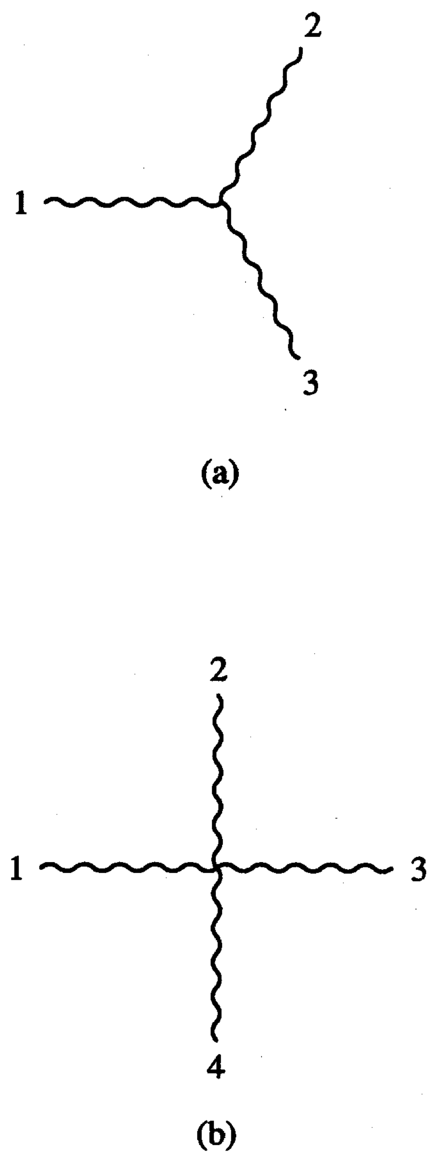

Fig. 1. Basic vertices for zero-mode diagrams.